Physics of complex systems and criticality

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1st Summer School of Interdisciplinary Research on Brain Network Dynamics, Terzolas, June 24-28, 2019
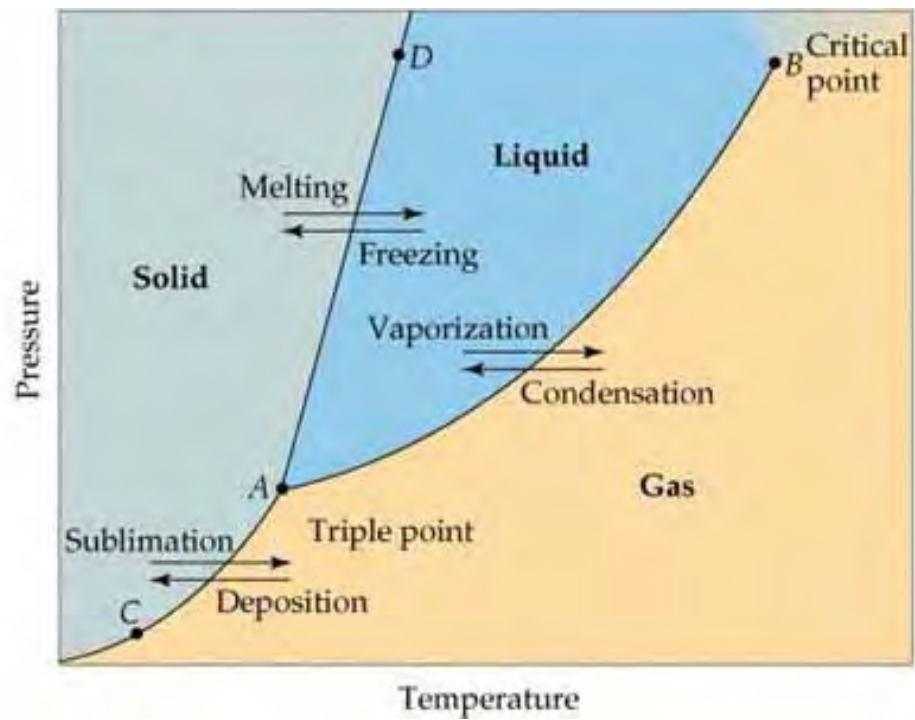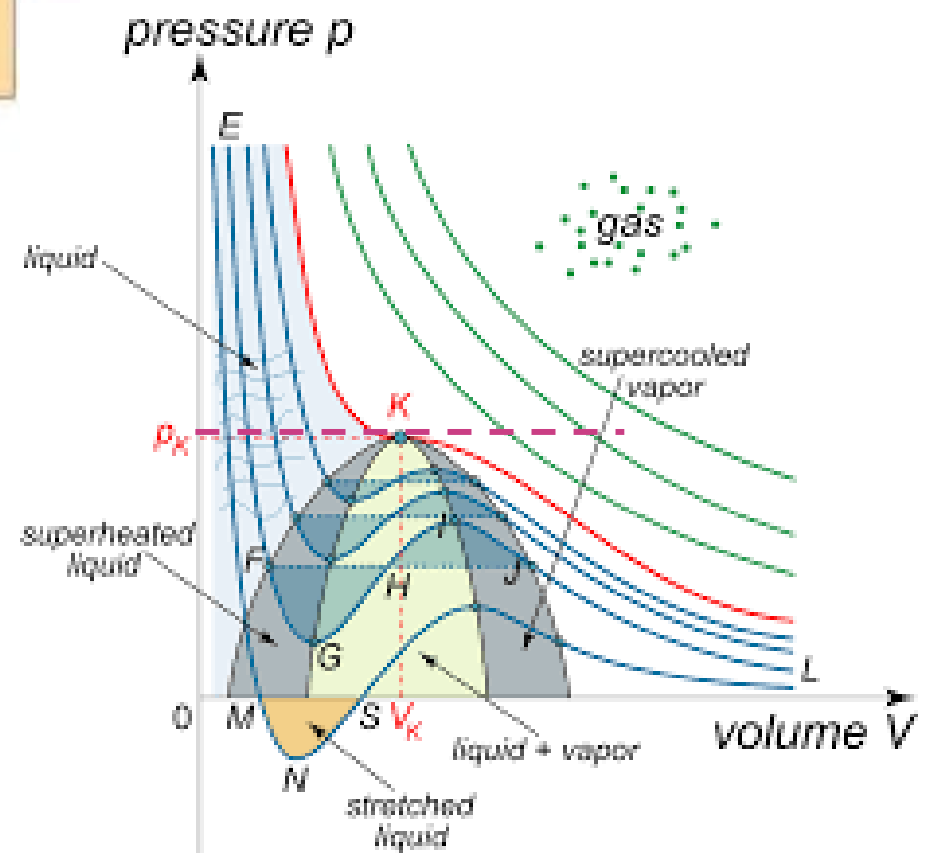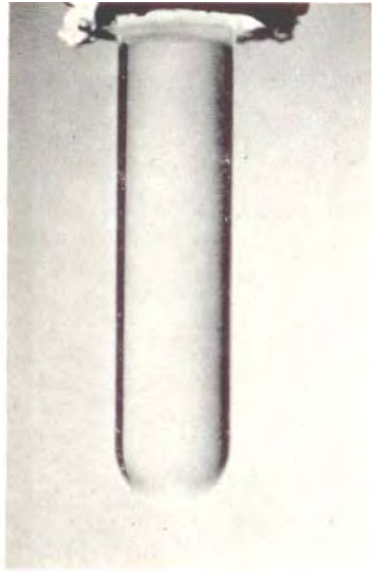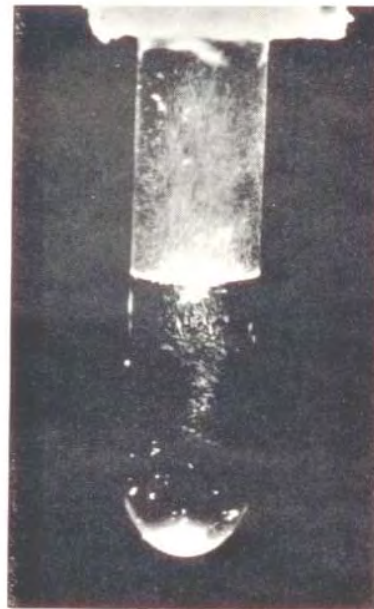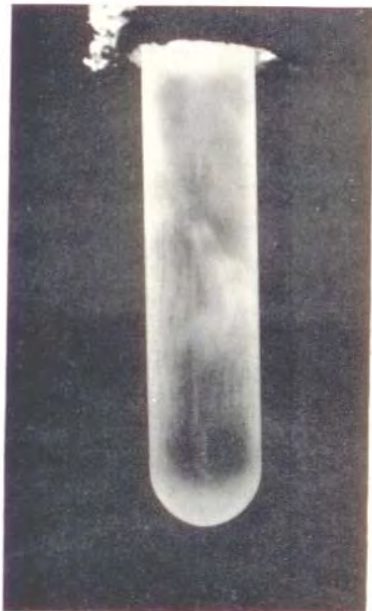
Diagram PT

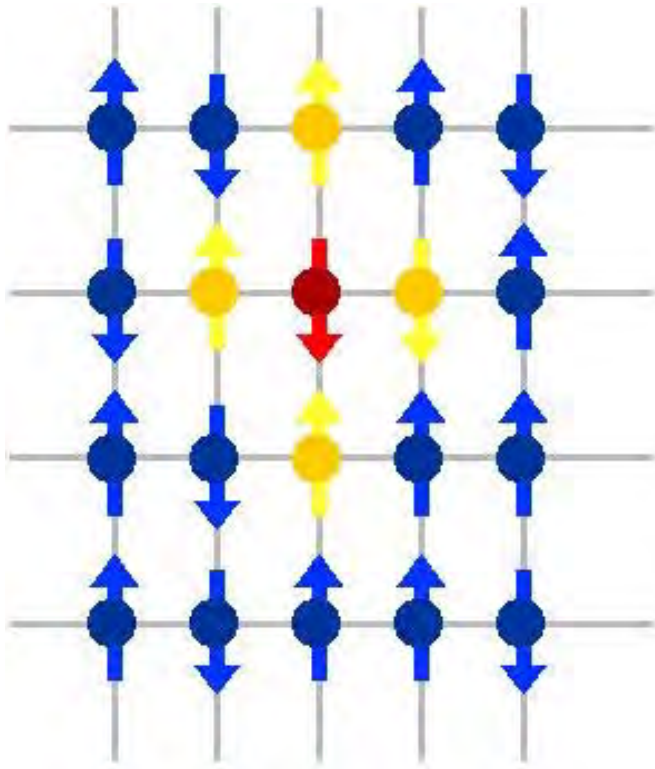Diagram PV

$T \gg T_c$

critical opalescence

$T \approx T_c$

$T \ll T_c$

a

b

c

What is a critical state?

- Ising model: two-state spins
- Ferromagnetic interaction: the energy is lower if spins are parallel
- System evolves toward equilibrium state which minimizes the energy
- New state of a spin is assigned according to the probability

$$\propto e^{-\frac{E}{K_B T}}$$

- At $T \cong \infty$ all spins are uncorrelated
- At $T = 0$ all spins tend to allign

Emergence of spontaneous magnetization at Tc (h=0)

Symmetry breaking

$$H_{Ising} = -J \sum_{<ij>} S_i S_j - h \sum_i S_i$$

For *h=0* Hamiltonian of the Ising model is symmetric for spin inversion

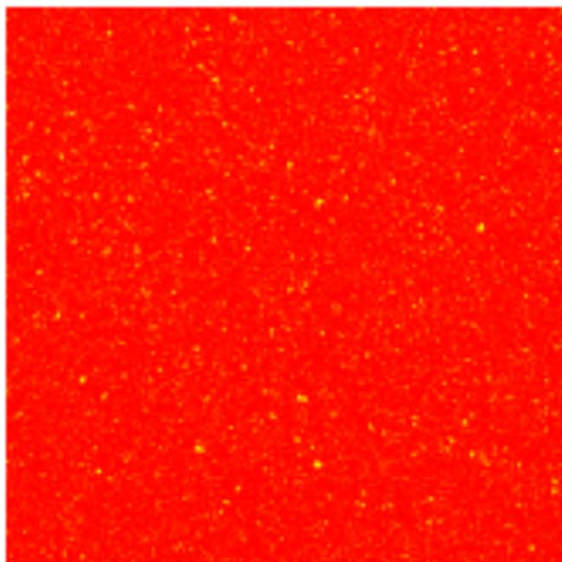Configurations with magnetization *+m* and *−m* have the same probability

$\longrightarrow$ <m>=0   no spontaneous magnetization!

In the limit $\boxed{N \to \infty}$ the symmetry between the two phases
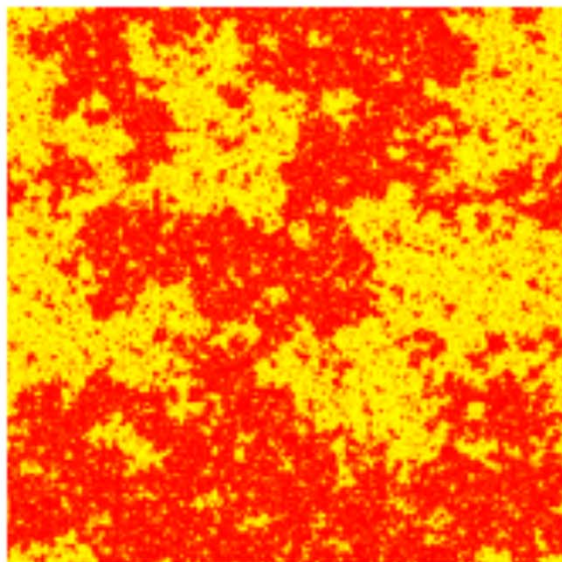can be broken and a spontaneous magnetization emerges

The system is trapped in a phase space subregion since the energy barrier
is too high

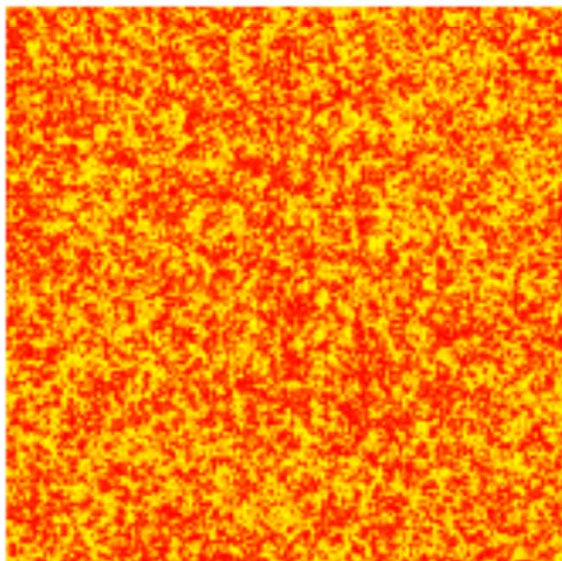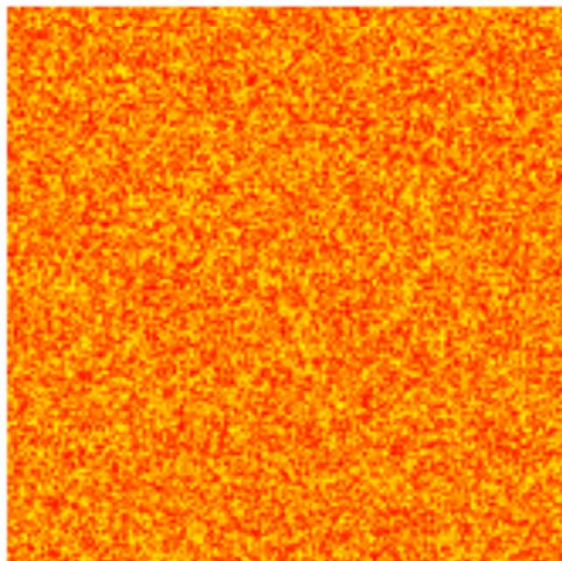Probability to escape $p \propto e^{-N^{(d-1)/d}}$

$T \approx 0$

$T \approx T_c$

$T \geq T_c$

$T \to \infty$

- clusters of all possible sizes are present

- divergence of correlation range

- divergence of fluctuations

- Self-similarity $\longrightarrow$ the largest cluster is fractal
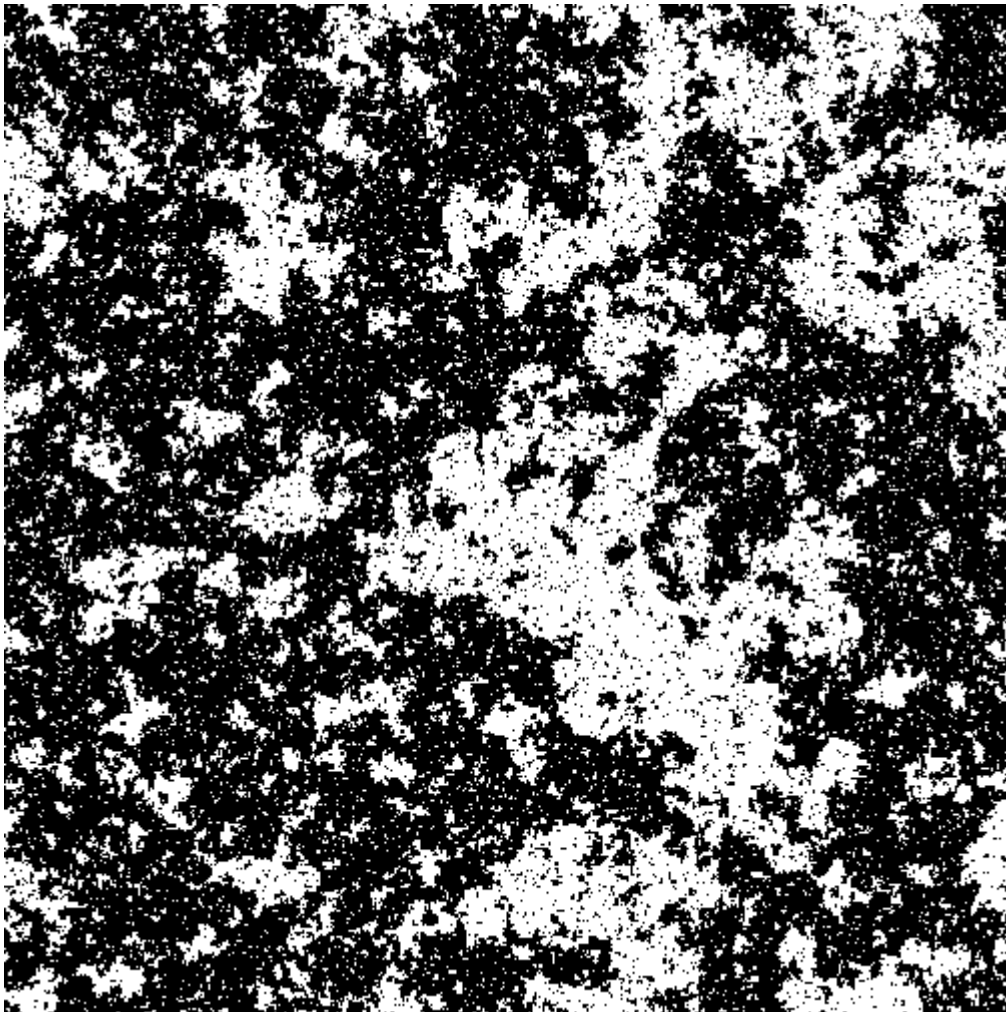
At the critical point

physical properties behave as

power laws:

Order parameter

Response function

Specific heat

....

Scale invariance – Mandelbrot set

SELF-SIMILARITY

Salvador Dalì "The war's face"

A part looks like the whole....

a

b

c

Self-similarity
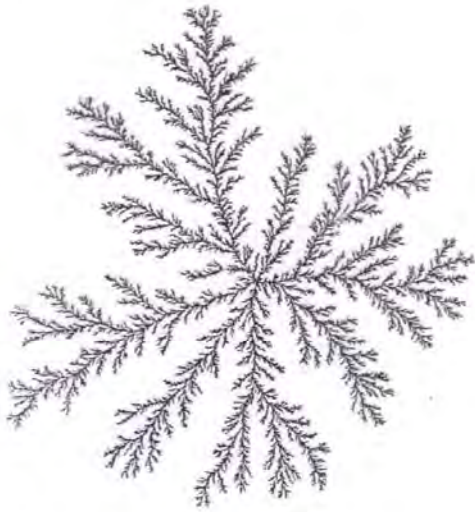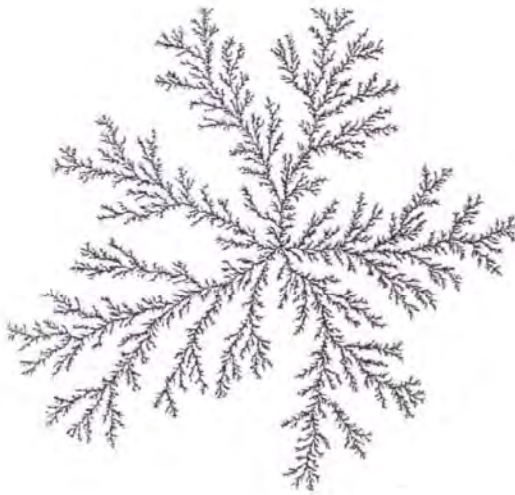
Diffusion Limited Aggregation

Correlation function

The correlation function

$$G(r) = <m(r)m(0)> - <m(0)>^2$$

measures the correlations between fluctuations in 0 and r

The spatial extent of the correlations is called **correlation length** ξ

The spatial integral of the correlation function provides the response function (derived by the fluctuation dissipation theorem)

$$\chi = \frac{1}{k_B T} \int d^d r \, G(r)$$

At the critical point $\xi \to \infty$

$$G(r) = G_0 \frac{k_B T}{\gamma} r^{-(d-2+\eta)} \exp(-r/\xi)$$

Experimentally can be measured by scattering experiments

Structure factor

$$S(q = 0) \propto \tilde{G}(0) = \int G(\vec{r}) d\vec{r}$$
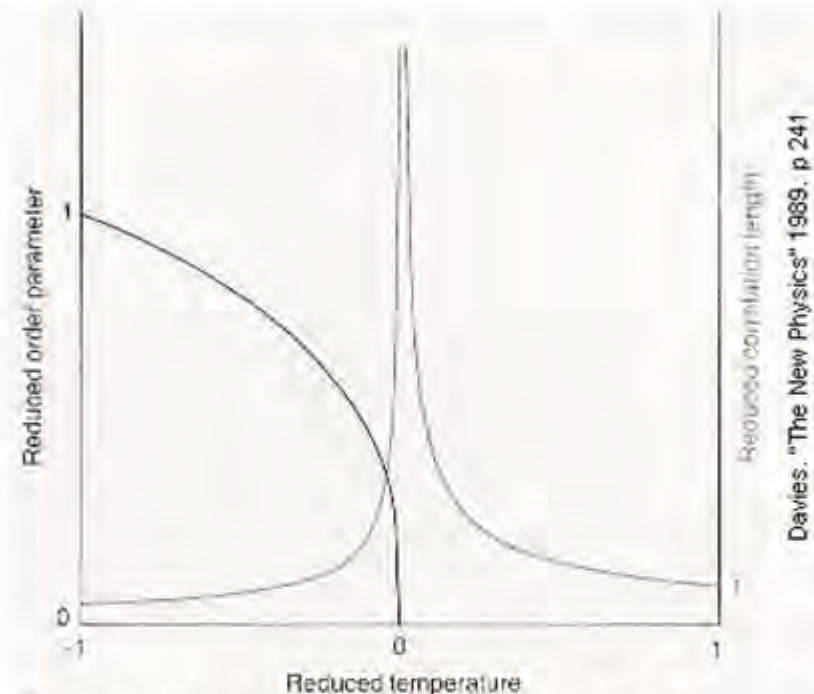
Critical Exponents

Reduced Temperature, t

$$t \equiv \frac{T - T_C}{T_C}$$

Specific heat $\quad C \propto |t|^{-\alpha}$

Magnetization $\quad M \propto |t|^{\beta}$

Magnetic susceptibility $\quad \chi \propto |t|^{-\gamma}$

Correlation length $\quad \xi \propto |t|^{-\nu}$

Critical behavior of the order parameter an the correlation length. The order parameter vanishes with the power β of the reduced temperature t as the critical point is approached along the line of phase coexistance. The correlation length diverges with the power ν of the reduced temperature.

The exponents display critical point universality (don't depend on details of the model).

Universality classes!

Universality
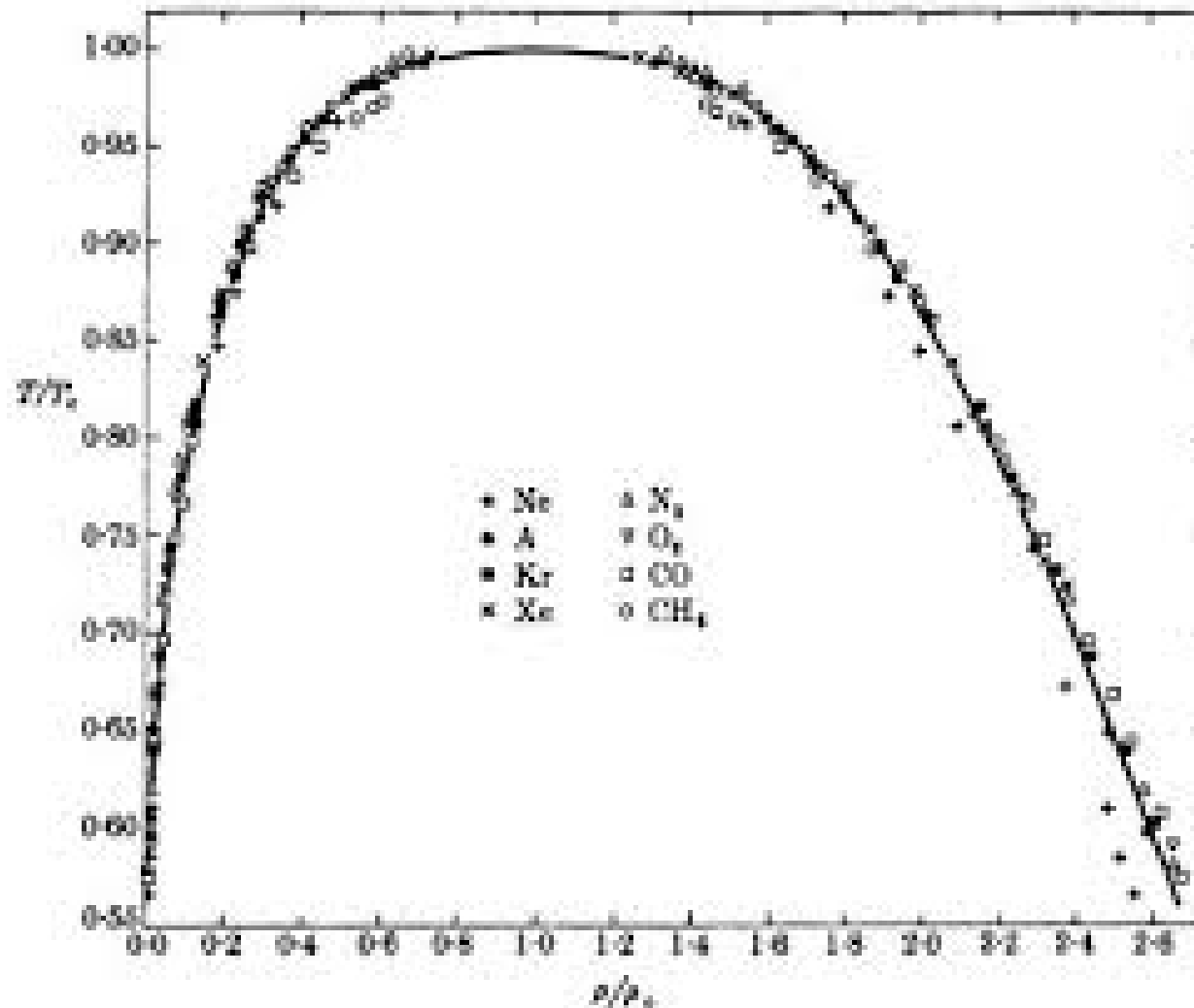
Densities of coexisting liquid and gas phases of a variety of substances, plotted against temperature, with both densities and temperatures scaled to their value at the critical point
(From:E A Guggenheim,J. Chem.Phys.,Vol.13, 253, 1945)

Exponents are not all independent….

$$\alpha + 2\beta + \gamma = 2,$$

$$\gamma = \beta(\delta - 1),$$

$$2 - \alpha = \nu d,$$

$$\frac{\gamma}{\nu} = 2 - \eta.$$

A number of relations have been derived among them

PROBLEM: thermodynamics predicts them as
INEQUALITIES
Whereas experimental values satisfy
EQUALITIES!!!

Scaling - Widom hypothesis

- Near the critical point the main physical properties exibit power law behaviour

- Nice properties of power laws $\longrightarrow$ invariant under rescaling!

Suppose $\qquad y = f(x) = x^{\alpha}$

Make the scale transformation

$$x \to x' = bx \qquad y \to y' = cy$$

Under rescaling

$$f(x) \to f(x') = cf(x'/b)$$

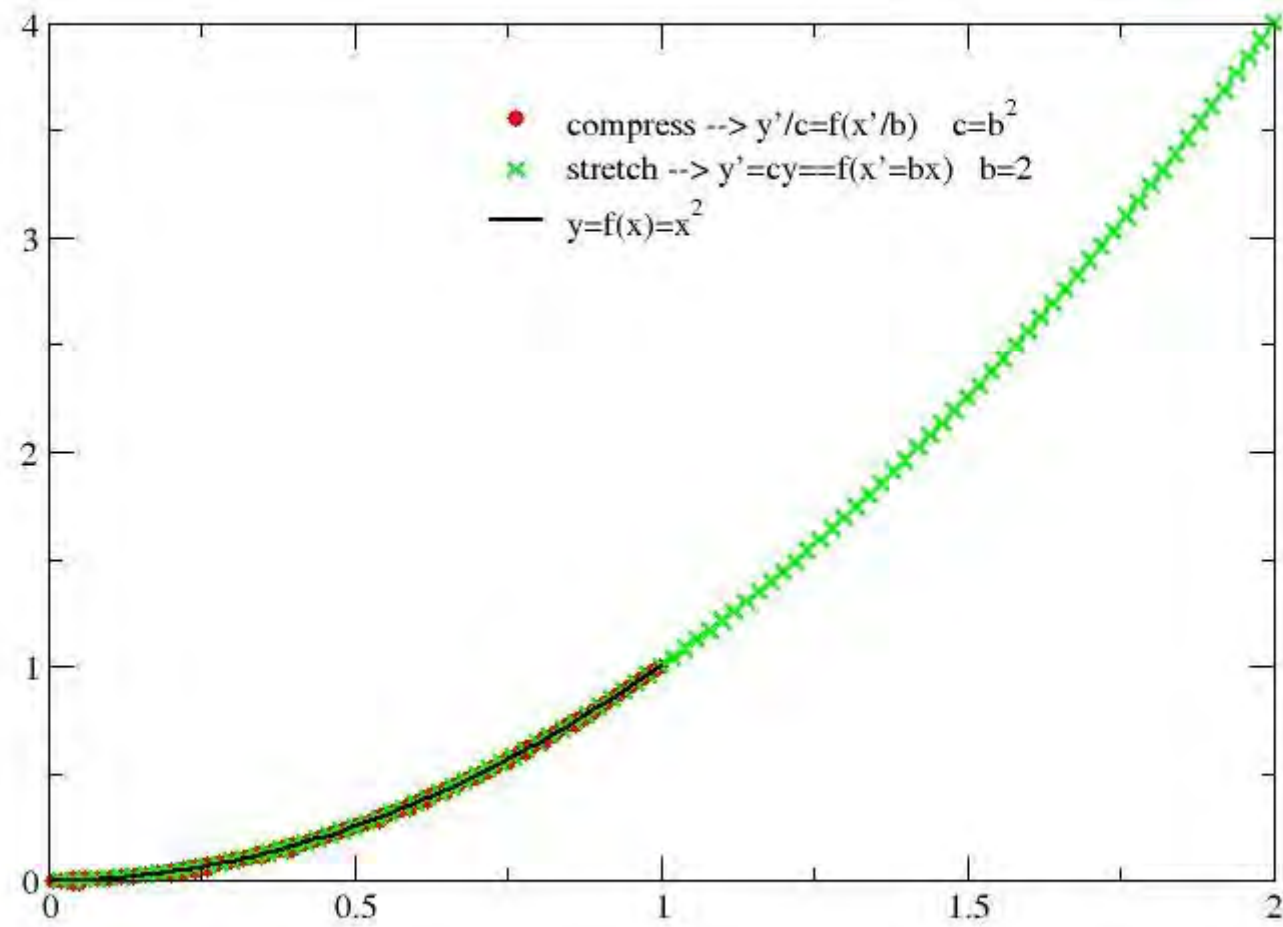If $\quad c = b^{\alpha} \qquad$ the function is invariant

$\longrightarrow$ **Scale invariance**

$$f(\lambda x) = g(\lambda)f(x)$$

homogeneous functions

Power law

Exponential function

Legend:
- compress --> $y'/c = f(x'/b)$ $c = e^b$
- stretch --> $y' = cy = f(x' = bx)$ $b = 2$
- $y = f(x) = e^x$
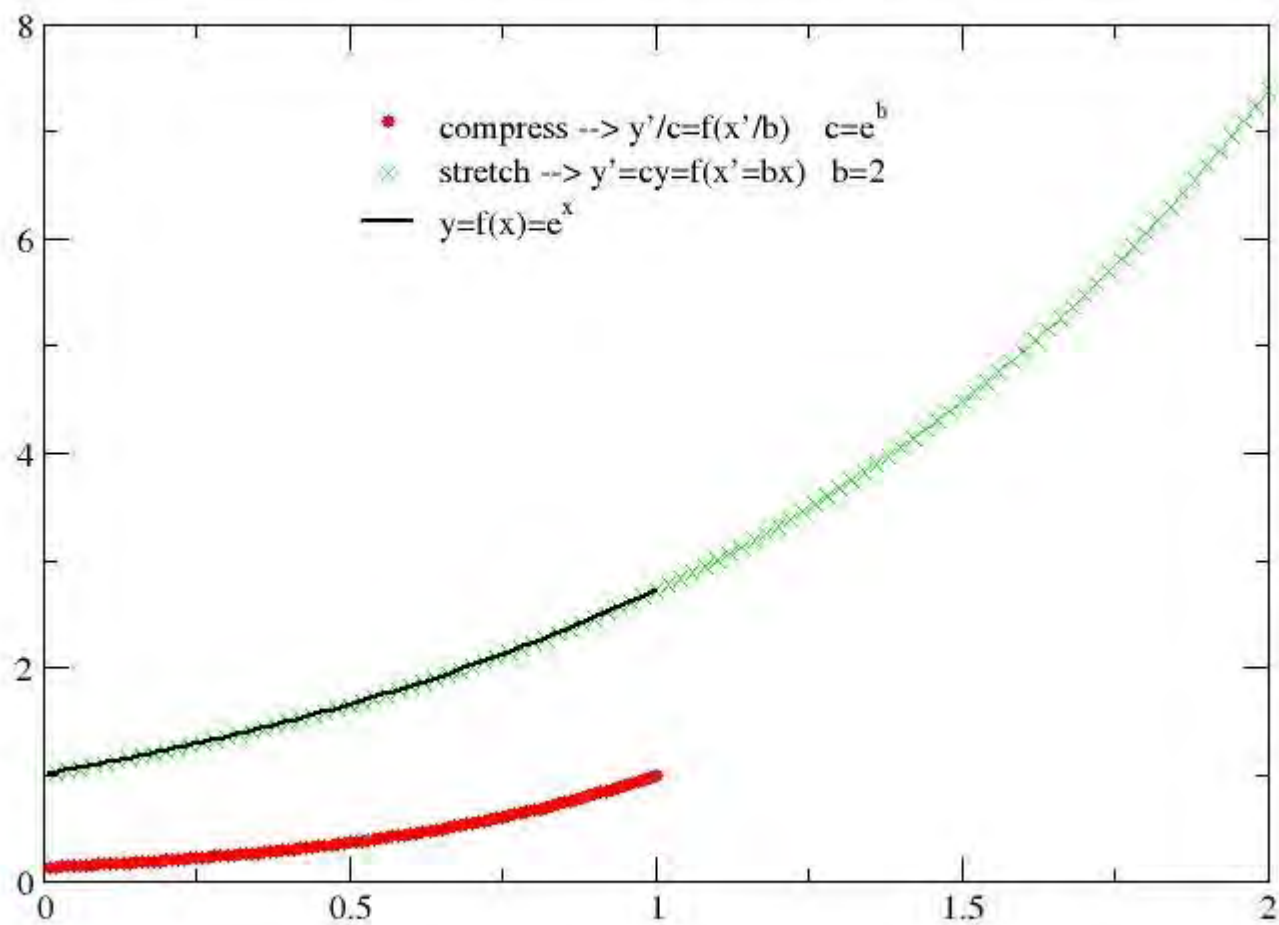
For functions of more than one variable

$$f(\lambda^a x, \lambda^b y) = \lambda f(x, y) \qquad \forall \lambda$$

Choose

$$\lambda = y^{-1/b}$$

Obtain

$$f(x, y) = y^{1/b} f(y^{-a/b} x, 1) = y^{1/b} g(y^{-a/b} x)$$

Where g is an universal function

Assuming that thermodynamic potentials are GHF leads to relations among critical exponents as equalities

What is a complex system?

- Many components or degrees of freedom

- Interactions among components $\Longrightarrow$ cooperative effects
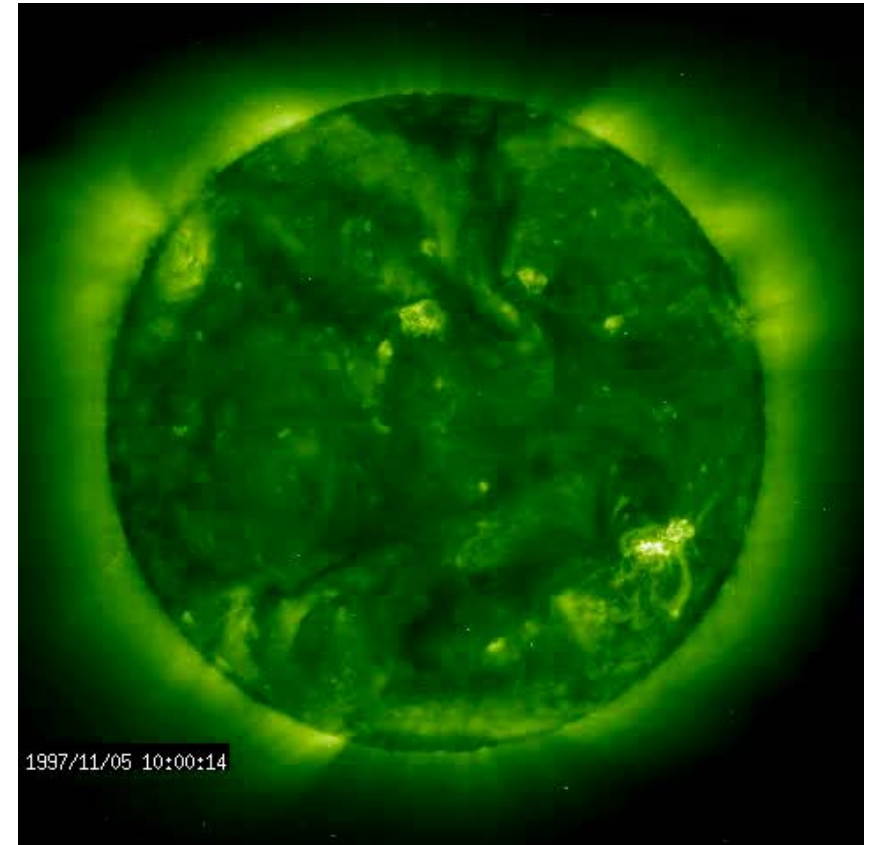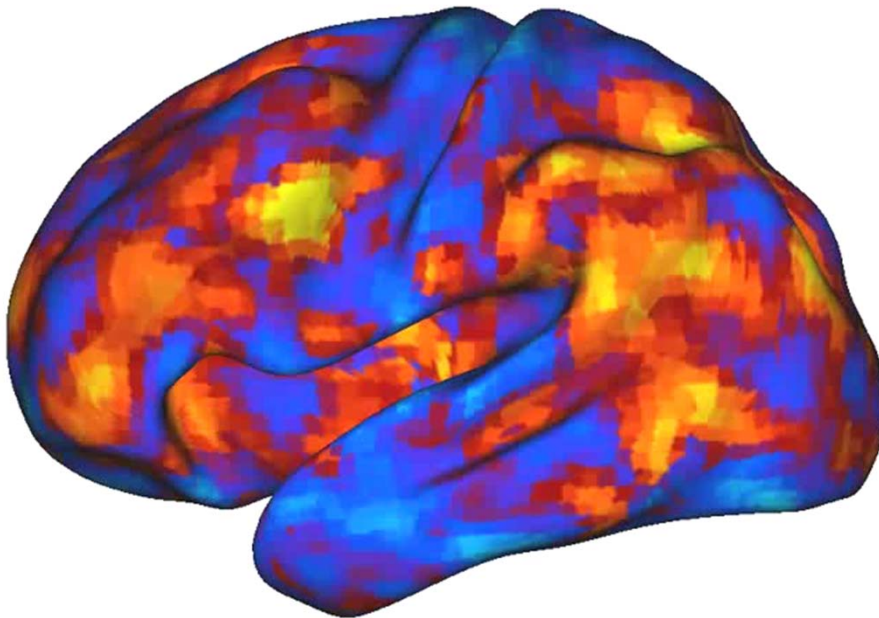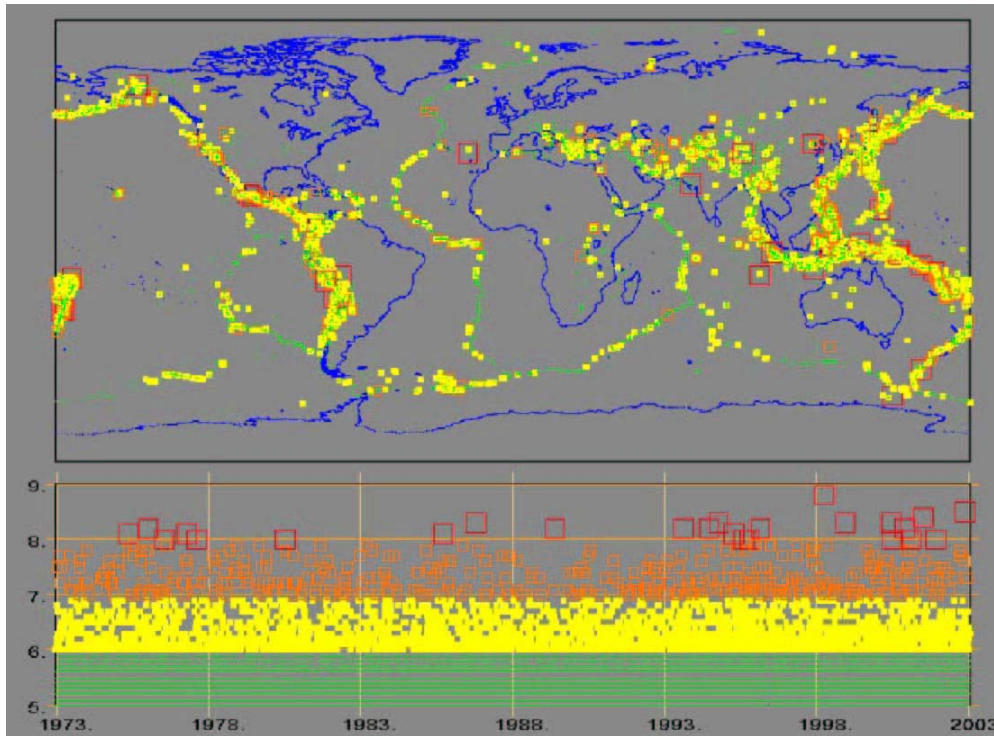
- Emergence of «impredictable» macroscopic behaviour
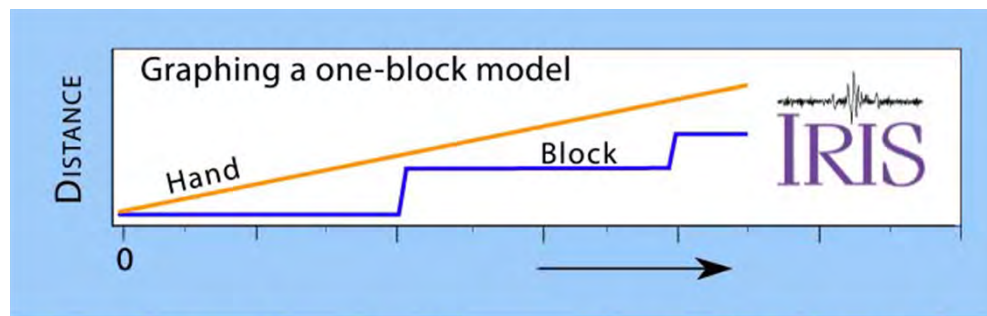
SELF-ORGANIZATION

What is the signature of a complex system?

- Fundamental properties exhibit singular behaviour

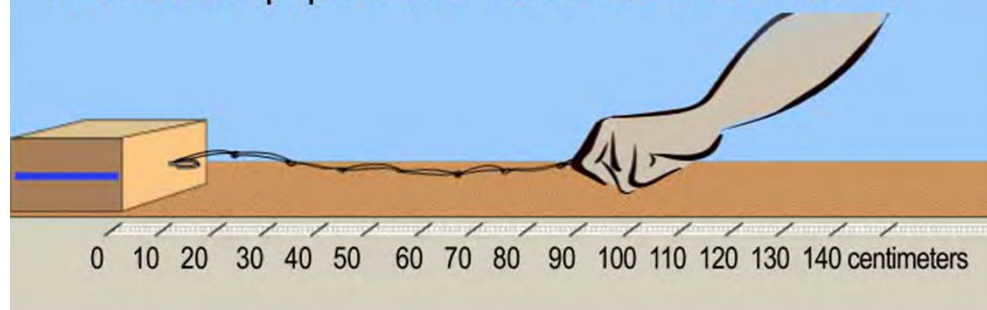- Emergence of power laws

- Absence of a characteristic scale

"I think you should be more explicit here in step two."

Complex systems

Graphing a one-block model

Hand
Block

IRIS

DISTANCE

0

Earthquake Machine—Elastic Rebound

Block & sandpaper models strain & elastic rebound

0 10 20 30 40 50 60 70 80 90 100 110 120 130 140 centimeters

A single block pulled on a rough surface slips always the same distance

$$\Delta x = (\mu_s - \mu_d)(N/K)$$

On real faults earthquakes of all size are measured

moving plate

K_L

K

K

fixed plate

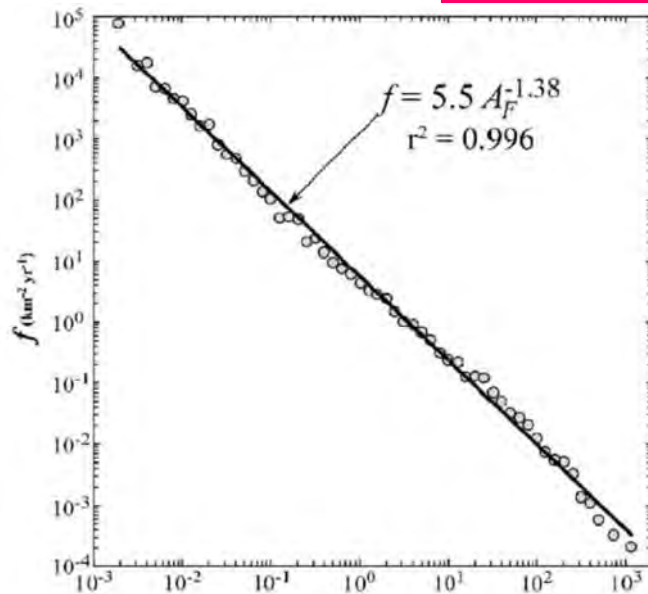worldwide 1973-2002
S. California 1984-2001

M
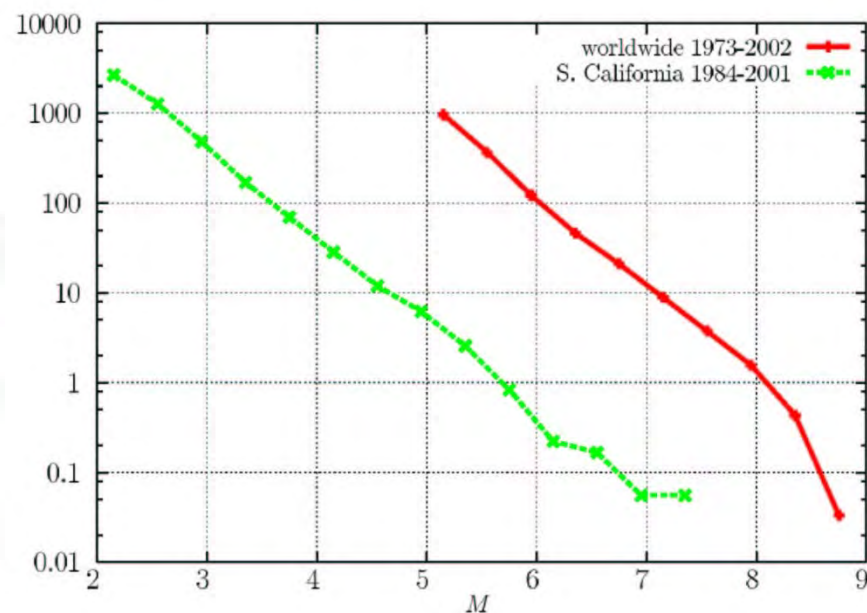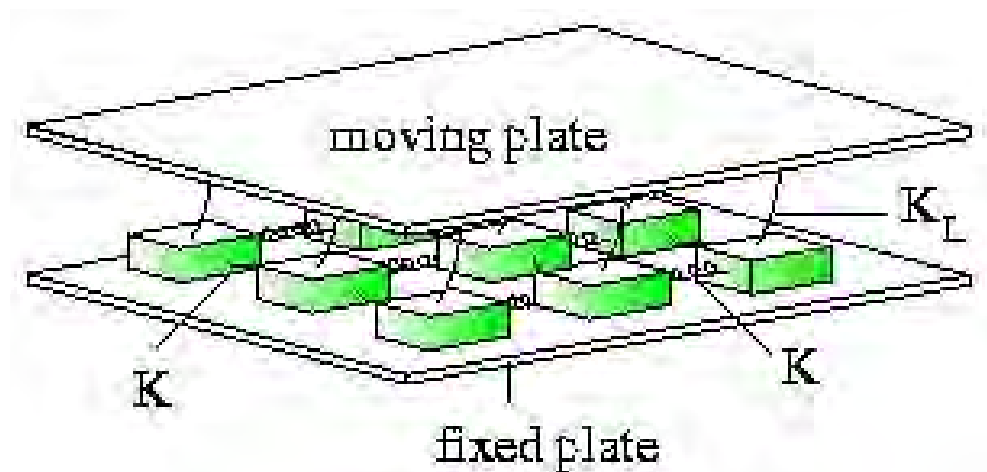
From complexity

Bull by Picasso

To Universality

Power laws in nature

Forest fires in Ontario (Canada) 1976-1996
Turcotte & Malamud 2004

$f = 5.5\, A_F^{-1.38}$
$r^2 = 0.996$

Rockfall in Umbria (1997)
& Yosemite (1980-2002)
Malamud 2004

Areas covered by lava
in volcanic eruptions
(Springerville,
Arizona) Lahaie &
Grasso 1998

Exponent 1.1

$R^2=0.998$
$D=1.15$

- Earthq. California Cat.
- Solar Flares GOES
- Solar Flares BATSE
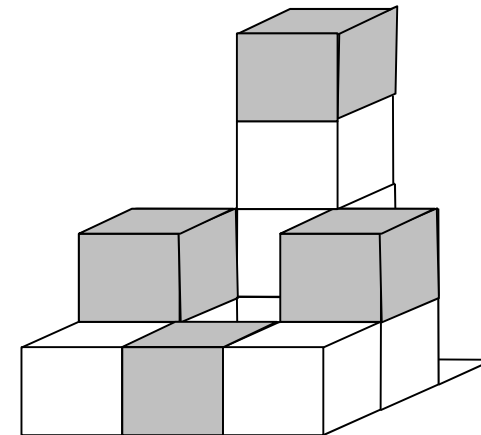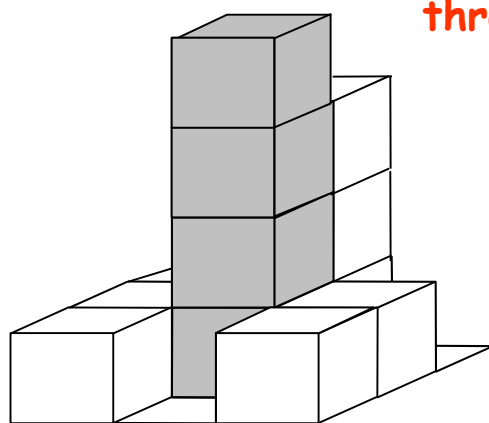- Solar Flares WATCH

$\alpha=1.65\pm0.1$
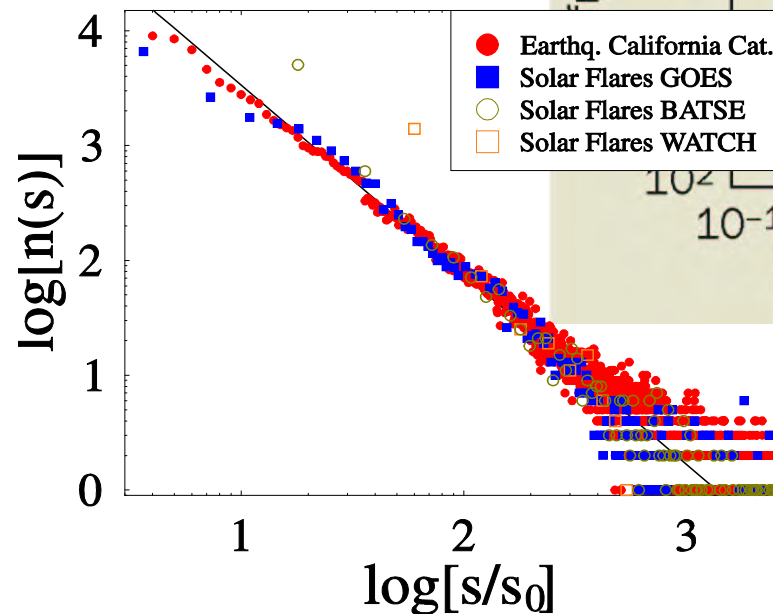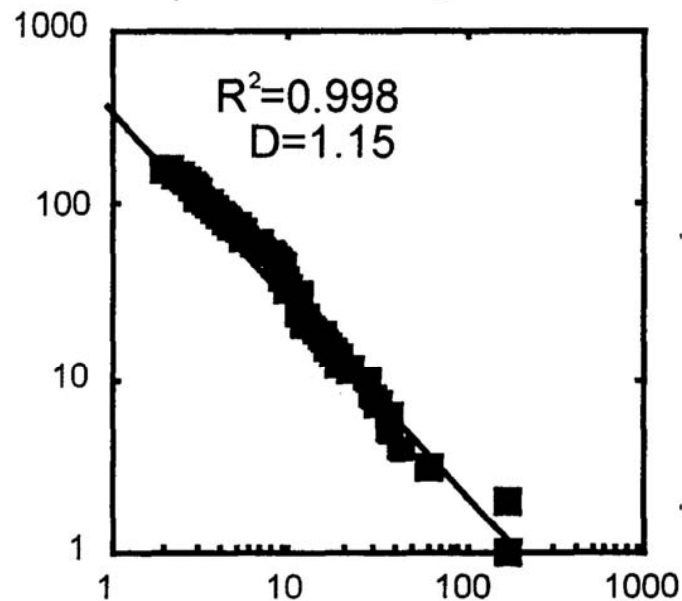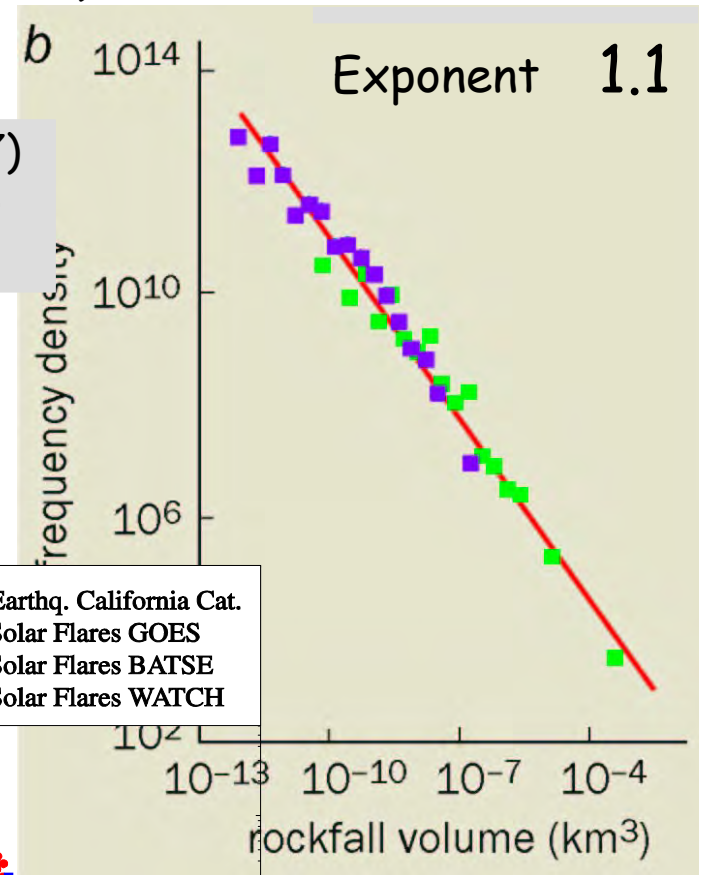
SELF-ORGANIZED CRITICALITY

Bak, Tang, Wiesenfeld, PRL 1987

Dynamical systems spontaneously evolving toward a critical state
without parameter tuning ⟶ <u>no characteristic event size</u>

Sand pile   by adding at random one grain…

threshold=4

Size and duration distribution

$P(s) \sim s^{-1}$

$P(T) \sim T^{-0.5}$

Fundamental ingredient: separation of time scales

- **Slow scale: adding a grain**
- **Fast scale: propagation of an avalanche**

SOC applied to many natural phenomena

❖ Slides and avalanches

❖ Neural activity

❖ Solar flares

❖ Fluctuations in confined plasma

❖ Biological evolution

❖ Earthquakes

- The process generated by the sandpile or other standard SOC models is
Poissonian ⟶ absence of temporal correlations

- Additional ingredients must be introduced to generate a correlated process

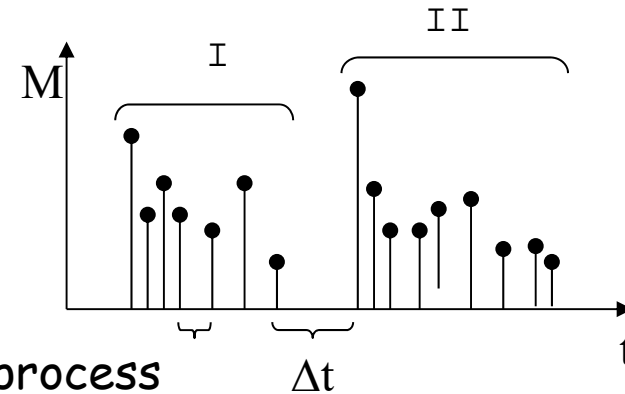- In many stochastic processes in nature temporal correlations are present
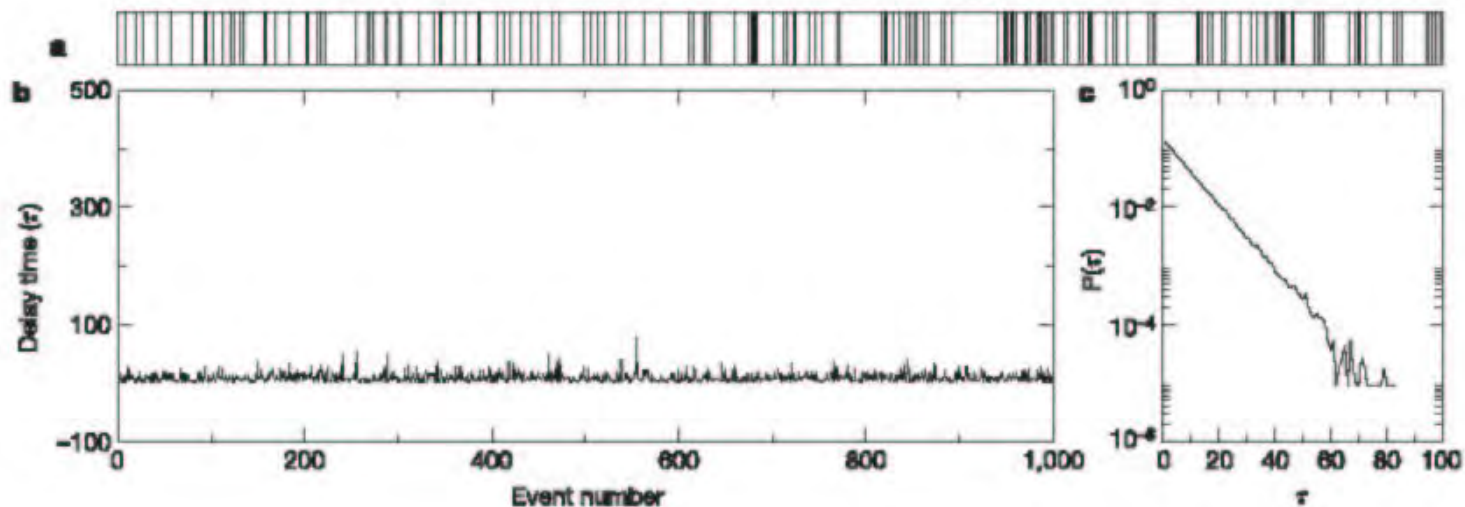
- How can we detect them?

Probability distribution of intertimes

$$\Delta t$$

between consecutive events

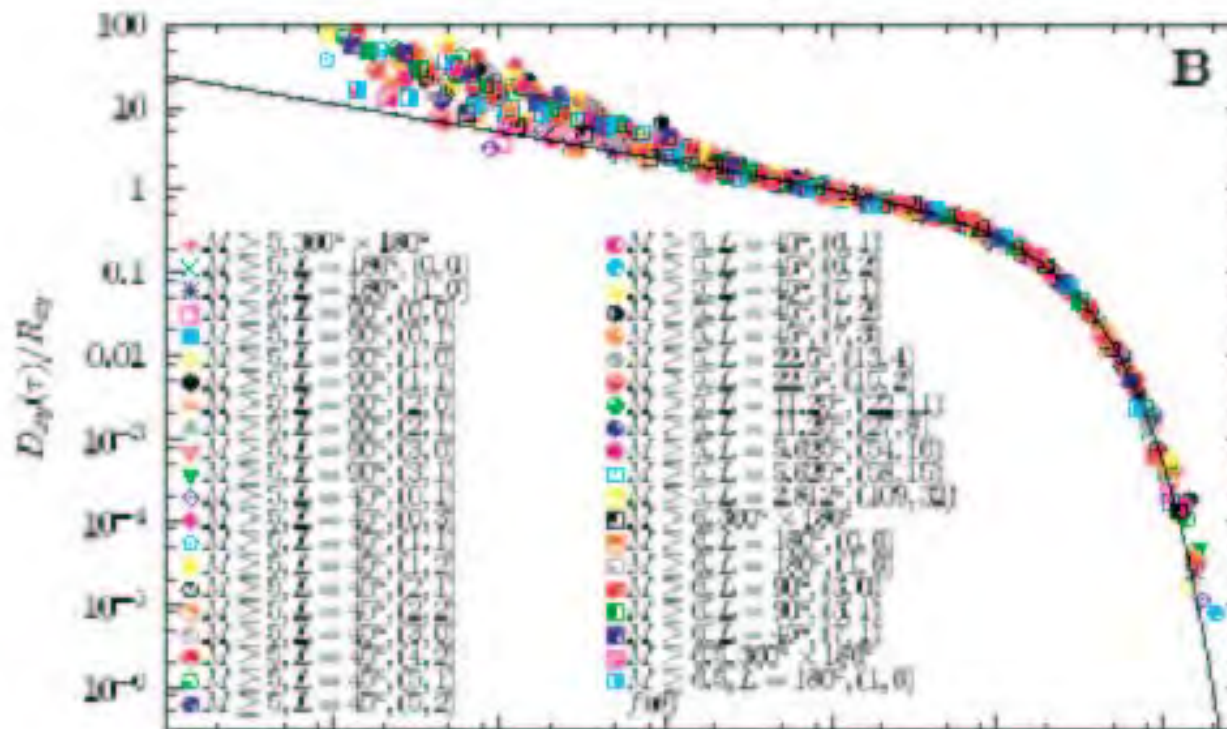- $P(\Delta t)$ is an exponential for a Poisson process

Barabasi, Nature 2005

- It exhibits a more complex structure as temporal correlations are present in the process

⇒ Corral (PRL, 2004) rescaling Δt by the average rate in the area
obtained a universal scaling law for the probability density

$$D(\Delta t, M_c) = R(M_c) f(R(M_c) \Delta t)$$

holds also for Japan, Spain, New Zeland...
scaling function not universal
(different areas are characterized by different rates)

Wiener – Khintchine Theorem

Autocorrelation function $K(s) = K(t_2 - t_1) = <A(t_2)A(t_1)>$

Power spectrum

$$S(f) = 4\int_0^\infty ds K(s) \cos(2\pi f s)$$

If the variable is very
irregular (unpredictable)
Then $K(s)=c\delta(s)$
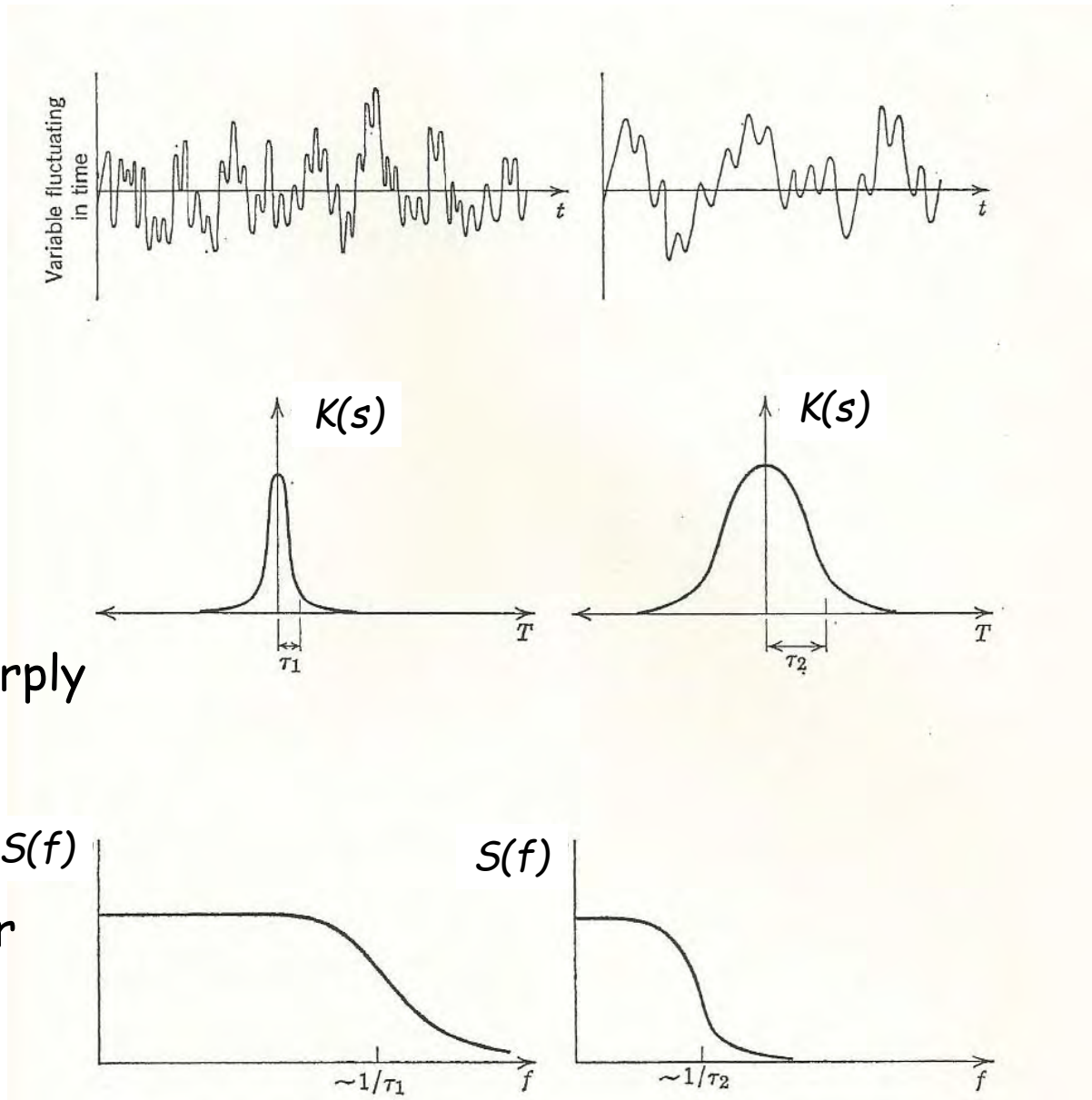and
$S(f)=2c$ for all f

$\longrightarrow$ *White noise*

but $K(0)$ would diverge!

In reality $K(s)$ decays sharply
within τ

Then $S(f)$ is constant over
a frequency range $1/\tau$
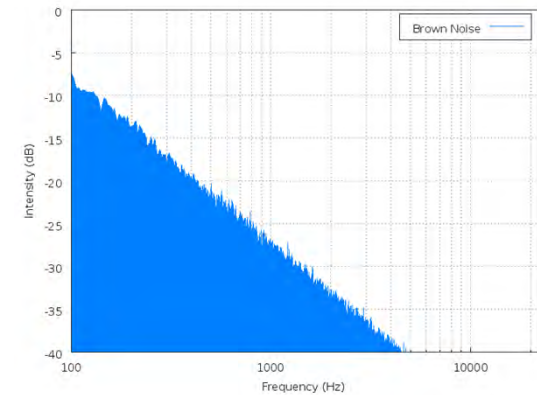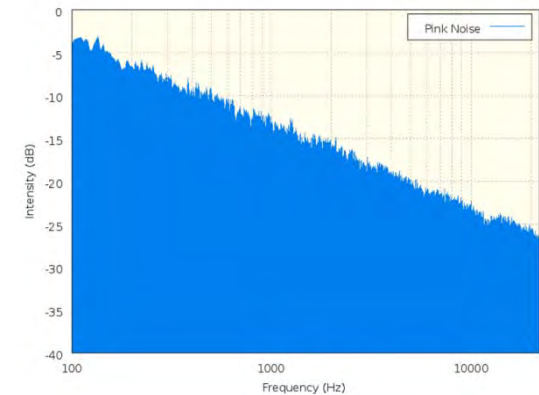
Color of noise

When not white noise is colored

➡ power law behaviour $f^{-\beta}$

Pink (flicker) $\beta=1$

Brown (red) noise $\beta=2$

(by integrating white noise)

f^{-1} noise ➡ long range temporal correlations

f^{-2} noise ➡ uncorrelated signal

Take-home message

- Criticality implies the absence of a characteristic scale

- Emergence of power laws can be explained by SOC

- Power law distributions are not sufficient for criticality

- Necessity to verify the existence of long-range temporal correlations